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ASYMPTOTIC BEHAVIOR FOR A STRONGLY DAMPED
NONLINEAR WAVE EQUATION[*]

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This paper is a specific application of the author's recent paper, on "Limiting Behavior for Strongly Damped Nonlinear Wave Equations" where results of Webb and Fitzgibbon were extended by applying results of a few recent papers written by the author. Some of the main results of this paper are to show boundedness of orbits in one space implies boundedness of orbits in other spaces (the technique ehre provides an interesting

alternative proof of the main results of Alikakos. Invariant
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and imply bounded dissipative in spaces of "smoother" functions,
the existence of a "very smooth" maximal compact invariant set
under a very weak dissipative assumption, along with its strong
stability and attractivity properties in several spaces, and
fixed point theorems under these weak dissipative hypotheses.

The first section will show existence, uniqueness and the variation
of constants formula, along with certain compactness properties of
the orbits for Equation (1). The second dection will discuss
boundedness of orbits. The third section will discuss invariant
sets. Finally, the fourth section will discuss the limiting behavior
under weak dissipative conditions.

ASYMPTOTIC BEHAVIOR FOR A STRONGLY DAMPED
NONLINEAR WAVE EQUATION

Paul Massatt

In this paper, we consider the limiting behavior of equations of the form

$$u_{tt} - \alpha\Delta u_t - \Delta u = \mathscr{F}(t,x,u,\nabla u,u_t,\nabla u_t) \qquad (1)$$

with $\alpha > 0$ and $\mathscr{F}$ periodic in t. We also consider special cases where $\mathscr{F}$ may depend on fewer terms. We are interested in solutions of the form $u(t,x): \mathbb{R}^+ \times \Lambda \to \mathbb{R}$ where Λ is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $u + \beta \frac{\partial u}{\partial \nu} = 0$ for all $x \in \partial\Lambda$ and some $\beta \geq 0$, and $u(0,x) = \phi(x)$ and $u_t(0,x) = \psi(x)$ for $\phi(x), \psi(x)$ in some appropriate space of initial functions. The case of Neumann boundary conditions could also be considered but is slightly more complicated and so we have omitted this case from discussion in this paper. (See [14]).

This paper is a specific application of my recent paper, "Limiting Behavior for Strongly Damped Nonlinear Wave Equations" [14] where results of Webb [16] and Fitzgibbon [4] were extended by applying results of a few recent papers of mine ([11], [12], [13], [14]). I am deeply grateful to Professor Jack K. Hale for his help and supervision in the preparation of this paper.

Before describing the results, a few definitions will be needed. Let $T: X \to X$ be a continuous map on a Banach space X. A bounded set $B \subset X$ dissipates a set J if there is an integer $n_0 > 0$ such that $T^n(J) \subset B$ for all $n \geq n_0$. T is point dissipative if there is a bounded set B which dissipates all points in X. T is compact dissipative if there is a bounded set B which

dissipates all compact sets in X. T is <u>local dissipative</u> if there is a bounded set B which dissipates a neighborhood of any point in X. T is <u>local compact dissipative</u> if there is a bounded set B which <u>dissipates</u> a neighborhood of any compact set in X. T is <u>bounded dissipative</u>, or <u>ultimately bounded</u> if there is a bounded set B which dissipates all bounded sets in X.

Some of the main results of this paper are to show boundedness of orbits in one space implies boundedness of orbits in other spaces (the technique here provides an interesting alternative proof of the main results of Alikakos in [1], see [14]), invariant sets in one space are automatically invariant sets in many spaces (which implies smoothness properties of invariant sets), point dissipative and compact dissipative are equivalent in many spaces and imply bounded dissipative in spaces of "smoother" functions, the existence of a "very smooth" maximal compact invariant set under a very weak dissipative assumption, along with its strong stability and attractivity properties in several spaces, and fixed point theorems under these weak dissipative hypotheses.

To understand these results we introduce some definitions. The map T may be thought of as the period map. A fixed point, then, corresponds to a periodic solution. The <u>orbit</u> of a set B under T, $\gamma^+(B)$, is defined by $\gamma^+(B) = \bigcup_{n=0}^{\alpha} T^n(B)$. The ω-limit set of B, $\omega(B)$, is defined by $\omega(B) = \bigcap_{m=0}^{\infty} Cl\{\bigcup_{n=m}^{\infty} T^n(B)\}$. A set J is invariant under T if $TJ = J$. Let $Ba(x)$ be a ball of radius a centered at x. A set J is <u>stable</u> if, for all $\epsilon > 0$, there is a $\delta > 0$ such that for all $n \geq 0$, $T^n(J + B_\delta(0)) \subset J + B_\epsilon(0)$. J attracts B if every neighborhood of J dissipates B. J is <u>uniformly asymptotically stable</u> if J is stable and attracts a neighborhood of itself.

The first section of the paper will show existence, uniqueness, and the variation of constants formula, along with certain compactness properties of the orbits for Equation (1). The second section will discuss boundedness of orbits.

The third section will discuss invariant sets. Finally, the fourth section will discuss the limiting behavior under weak dissipative conditions.

1. <u>Existence, Uniqueness, and Compactness Results.</u> Equation (1) may also be considered as an ordinary differential equation on a Banach space. This is the setting I prefer, as it usually seems much more convenient. If $v = u_t$, then

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{bmatrix} 0 & 1 \\ \Delta & \alpha\Delta \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{F}(t,x,u,\nabla u,v,\nabla v) \end{pmatrix} \qquad (2)$$

or

$$z_t = -Bz + F(t,z) \quad \text{where} \quad B = \begin{bmatrix} 0 & -1 \\ -\Delta & -\alpha\Delta \end{bmatrix}, \quad z = \begin{pmatrix} u \\ v \end{pmatrix},$$

$$F(t,z) = \begin{pmatrix} 0 \\ \mathcal{F}(t,x,u,\nabla u,v,\nabla v) \end{pmatrix}, \quad \text{and} \quad z_0 = \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

We may consider a variety of spaces, such as $L^p \times L^p$, $\overset{*}{W}{}^{1,p} \times L^p$ (where $\overset{*}{W}{}^{m,p}$ is the completion in $W^{m,p}$ of C^m functions with the appropriate boundary conditions). $\overset{*}{W}{}^{2,p} \times L^p$, or $\overset{*}{W}{}^{m,p} \times \overset{*}{W}{}^{n,p}$ where $1 \leq p \leq \infty$ and $1 \leq n \leq m \leq 2$.

Following the style of D. Henry [9], we show that the solution map to

$$z_t = -Bz, \qquad (3)$$

sometimes denoted e^{-Bt}, is an analytic semigroup. An analytic semigroup on a Banach space X is a family of continuous linear operators on X, $\{T(t)\}_{t \geq 0}$ satisfying:

(i) $T(0) = I$, $T(t)T(s) = T(t+s)$ for $t, s \geq 0$

(ii) $T(t)x \to x$ as $t \to 0^+$, for each $x \in X$

(iii) $t \to T(t)x$ is real analytic on $0 < t < \infty$ for each $x \in X$.

The infinitesimal generator L of this semigroup is defined by
$Lx = \lim_{t \to 0^+} \frac{1}{t}(T(t)x-x)$, its domain $D(L)$ consisting of all $x \in X$ for which
this limit exists. We usually write $T(t) = e^{Lt}$.

A linear operator A in a Banach space X is called a <u>sectorial operator</u>
if it is a closed, densely defined operator such that, for some ϕ in $(0, \pi/2)$
and some $M \geq 1$ and real a, the sector $S_{a,\phi} = \{\lambda | \phi < |arg(\lambda-a)| \leq \pi, \lambda \neq a\}$
is in the resolvent set of A and $||(\lambda-A)^{-1}|| \leq M/|\lambda-a|$ for all $\lambda \in S_{a,\phi}$.

It is known that if A is sectorial, -A generates an analytic semigroup
e^{-At} (see [9]), and conversely, if -A generates an analytic semigroup, then
A is sectorial. Furthermore, $e^{-At} = \frac{1}{2\pi i}\int_{\Gamma}(\lambda+A)^{-1}e^{\lambda t}d\lambda$, where Γ is a contour
in $\rho(-A)$ with $\arg \lambda \to \pm\theta$ as $|\lambda| \to +\infty$ for some $\theta \in (\pi/2, \pi)$. If
Re $\sigma(A) > \delta$, then $||e^{-At}|| \leq Ce^{-\delta t}$ for all $t > 0$ and for some constant C.

We now apply these results to the operator B. We know Δ is a sectorial
operator on $L^p, \mathring{W}^{1,p}$, and $\mathring{W}^{2,p}$. The operator $-\Delta$ has simple eigenvalues
$\{\lambda_n\}_{n \geq 0}$ with $\lambda_n > 0$, $\{\lambda_n\} \to +\infty$, and corresponding eifenfunctions $\{\rho_n(x)\}$. By
using these eigenfunctions we can show B has eigenvalues $\{r_n\} \cup \{q_n\}$ where

$$r_n = \frac{\alpha\lambda_n}{2} - \frac{1}{2}\sqrt{\alpha^2\lambda_n^2 - 4\lambda_n} \quad \text{and} \quad q_n = \frac{\alpha\lambda_n}{2} + \frac{1}{2}\sqrt{\alpha^2\lambda_n^2 - 4\lambda_n} \; . \quad \text{We get}$$

$\lim_{n \to \infty}\{r_n\} = \frac{2}{\alpha}$ and $\lim_{n \to \infty}\{q_n\} = +\infty$. Hence, we know $\{r_n\} \cup \{q_n\} \cup \{\frac{2}{\alpha}\}$ is in the
spectrum of B, with $\{r_n\} \cup \{q_n\}$ in the point spectrum and $\{\frac{2}{\alpha}\}$ in the
continuous spectrum.

We next show that this is the entire spectrum of B, and that B is
sectorial. To do this we formally calculate the resolvent of -B, $(\lambda+B)^{-1}$.

We get

$$(\lambda+B)^{-1} = (\lambda^2 - (\alpha\lambda+1)\Delta)^{-1}\begin{bmatrix} \lambda-\alpha\Delta & 1 \\ \Delta & \lambda \end{bmatrix}.$$

Since all the operators commute, it is easy to see that the inverse exists for all $\lambda \notin \{r_n\} \cup \{q_n\} \cup \{\frac{2}{\alpha}\}$. Furthermore, by using the bounds on the resolvent $(\lambda-\Delta)^{-1}$ (since we know Δ is sectorial) it is very easy to show B is also sectorial. Hence, e^{-Bt} is an analytic semigroup. Since $\text{Re } \sigma(B) < 0$ we have $||e^{-Bt}|| \leq Ke^{-\delta t}$ for some $K, \delta > 0$.

In the following we will use the fact that $e^{-Bt} = \int_\Gamma (\lambda+B)^{-1}e^{\lambda t}d\lambda$ where Γ is a contour in $\rho(-B)$, the resolvent of $-B$, with $\arg \lambda \to \pm\theta$ for some $\theta \in (\pi/2, \pi)$. This tells us that $e^{-Bt}(v(x))$, considered as an operator on $v(x)$ maps bounded sets in L^p to bounded sets in $\overset{*}{W}{}^{2,p} \times \overset{*}{W}{}^{2,p}$.

We now return to equation (2). If $F(t,z)$ is Hölder continuous in t and $F(t,\cdot): X \to Y$ is locally lipschitz with $D(A)_x \hookrightarrow Y$ ($D(A)$ on the space X is continuously imbedded into Y) then, following D. Henry [9], we have existence, uniqueness, and the variation of constants formula,

$$z(t) = e^{-B(t-t_0)}z_0 + \int_{t_0}^t e^{-B(t-s)}F(s,z(s))ds. \tag{4}$$

This formula will constantly be used as we discuss further results.

For compactness properties, it is useful to use the notion of the Kuratowski measure of noncompactness, or α-measure. The α-measure is a map $\alpha: \mathscr{B} \to [0,\infty]$ where $\mathscr{B}$ is the collection of bounded sets in X with the property $\alpha(B) = \inf\{r/B$ can be covered by a finite collection of sets of diameter less than $r\}$. The α-measure of noncompactness has the following properties:

(i) $\alpha(B) = 0$ if and only if $Cl\ B$ is compact,

(ii) $\alpha(BUC) = \max[\alpha(B), \alpha(C)]$,

(iii) $\alpha(A+B) \leq \alpha(A) + \alpha(B)$, and

(iv) $\alpha(Cl\ co\ A) = \alpha(A)$.

T is called an $\underline{\alpha\text{-constraction}}$ if there exists a $k \in [0,1)$ such that for all $B \in \mathscr{B}$ we have $\alpha(TB) \leq k\alpha(B)$. Often in dynamical systems, we need to consider a more general class of operators, the weak α-contraction. T is a conditional $\underline{\alpha\text{-contraction}}$ if there exists a $k \in [0,1)$ such that for all B with $TB \in \mathscr{B}$ we have $\alpha(TB) \leq k\alpha(B)$. Similarly, T is $\underline{\text{conditionally completely}}$ $\underline{\text{continuous}}$ if for all $B \in \mathscr{B}$ with $TB \in \mathscr{B}$ we have TB is precompact. In all the spaces we have considered, and with $\mathscr{F}$ satisfying the conditions to be mentioned below, the operator $U(t,t_0)$ defined by $U(t,t_0)z_0 = \int_{t_0}^{t} e^{-B(t-s)} F(s,z(s))ds$ where $z(s)$ is the solution with initial condition $z(t_0) = z_0$, is conditionally completely continuous. This will follow from the fact that either $F(t,\cdot)$ is completely continuous, or that $e^{-Bt}\binom{0}{v(x)}$ maps bounded sets in L^p into bounded sets in $\mathring{W}^{2,p} \times \mathring{W}^{2,p}$ and satisfies $\left\| e^{-Bt}\binom{0}{v(x)} \right\|_{\mathring{W}^{2,p} \times \mathring{W}^{2,p}} \leq Ce^{-\delta t} \|v(x)\|_p$ for some $C, \delta > 0$. Also, since for each space there is a $C, \delta > 0$ such that $\|e^{-Bt}\| \leq Ce^{-\delta t}$ for any $t > 0$ there is an equivalent norm where e^{-Bt} is a contraction. Using this and property (iii) of the α-measure it is easy to see that for all $t > t_0$ there is an equivalent norm where $X(t,t_0)$, the solution map of (2), is a conditional α-contraction.

Some of the relevance of this is found in the following theorem.

__Theorem 1:__ __If__ T __is a conditional α-contraction and__ $\gamma^+(B)$ __is bounded, then__ $\gamma(B)$ __is nonempty, compact, and attracts__ B. __Furthermore, if__ B __is precompact,__ __then__ $\gamma^+(B)$ __is precompact.__

In order to synthesize the results, we will call $\mathscr{A}$ the class of admissible spaces. Three special classes of admissible spaces will be defined. The class may be enlarged if greater smoothness assumptions on $\mathscr{F}$ are assumed (see [14]).

Case (i): $F(t,z(t)) = (\mathscr{F}(t,x,u,\nabla u,v,\nabla v))$, $\mathscr{F}$ is Hölder continuous in t, locally lipschitz and there exists a continuous function $K(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ such that $|\mathscr{F}(t,x,u,\nabla u,v,\nabla v)| \leq K(t)(1+|u| + |\nabla u| + |v| + |\nabla v|)$. Then the following spaces are in class $\mathscr{A}$: $\overset{*}{W}{}^{1,p} \times \overset{*}{W}{}^{1,p}$, $\overset{*}{W}{}^{2,p} \times \overset{*}{W}{}^{1,p}$, $\overset{*}{W}{}^{2,p} \times \overset{*}{W}{}^{2,p}$ for all $1 \leq p \leq \infty$, $\overset{*}{C}{}^{1,\mu} \times \overset{*}{C}{}^{1,\nu}$ for $0 \leq \nu < \mu \leq 1$, $\overset{*}{C}{}^{2} \times \overset{*}{C}{}^{1,\nu}$ for $0 \leq \nu < 1$, and $\overset{*}{C}{}^{2} \times \overset{*}{C}{}^{2}$.

Case (ii): $F(t,z(t)) = (\mathscr{F}(t,x,u,\nabla u,v)$, $\mathscr{F}$ is Hölder continuous in t, locally lipschitz and there exists a continuous function $K(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ such that $|\mathscr{F}(t,x,u,\nabla u,v)| \leq K(t)(1+|u| + |\nabla u| + |v|)$. Then the class $\mathscr{A}$ consists of all the spaces mentioned in (i) and the following additional spaces: $\overset{*}{W}{}^{1,p} \times L^p$, $\overset{*}{W}{}^{2,p} \times L^p$, $\overset{*}{C}{}^{2} \times C$ for $0 \leq \mu < 1$, and $\overset{*}{C}{}^{1,\mu} \times C^{\nu}$ for $0 \leq \mu, \nu < 1$.

Case (iii): $F(t,z(t)) = (\mathscr{F}(t,x,u,v))$, $\mathscr{F}$ is Hölder continuous in t, locally lipschitz and there exists a continuous function $K(\cdot): \mathbb{R}^+ \to \mathbb{R}^+$ such that $|\mathscr{F}(t,x,u,v)| \leq K(t)(1+|u|+|v|)$.

Then the class $\mathscr{A}$ consists of all the spaces in (i) and (ii) and the following spaces: $L^p \times L^p$ for $1 \leq p \leq \infty$ and $C^{\mu} \times C^{\nu}$ for $0 \leq \nu < \mu < 1$.

2. **Boundedness of Orbits.** It is often of great interest to know whether boundedness of orbits in one space implies boundedness of orbits in other spaces. This question was discussed for a predator-prey problem of great generality by Alikakos [1]. The methods discussed here are different, and I believe of a simpler

nature than those found in Alikakos. The arguments use the fact that the dynamical system is defined on a variety of spaces, the structure of the variation of constants formula, and the repeated application of the Sobolev Imbedding Theorem.

Theorem 2: For Equation (2), if $\gamma^+(z_0)$ is a bounded orbit in some admissible space $X \in \mathscr{A}$, then $\gamma^+(z_0)$ is a bounded orbit in all admissible spaces Y for that equation provided $z_0 \in Y$.

The proof may be found in [14]. We give an indication of the proof for an example. Let $z_0 \in L^q \times L^q$ and $z(t)$ bounded in $L^p \times L^p$ with $p < q \leq \infty$. Let $z(t)$ be bounded in $L^p \times L^p$. $z(t) = e^{-B(t-t_0)} z_0 + U(t,t_0)z_0$, with $U(t,t_0)z_0$ bounded in $\overset{**}{W}{}^{2,p} \times \overset{**}{W}{}^{2,p}$. The Sobolev Imbedding Theorem implies for some p, with $p < p_1 \leq q$ we have $\overset{**}{W}{}^{2,p} \subset L^{p_1}$. Hence $U(t,t_0)z_0$ is bounded in $L^{p_1} \times L^{p_1}$, and thus, $z(t)$ is bounded in $L^{p_1} \times L^{p_1}$. Proceeding by induction we can show $z(t)$ is bounded in $L^q \times L^q$.

3. **Invariant Sets.** Here we show an invariant set in one space is a compact invariant set in all the admissible spaces.

Theorem 3: If J is an invariant set in an admissible space X for the period map T, then J is a precompact invariant set in all the admissible spaces for the period map T.

The proof may be found in [14]. Although the proof is similar to the proof of Theorem 2.1, the following lemma is needed to resolve a problem.

Lemma 4. Let $X \hookrightarrow Y$. Let $J \subset X$ be precompact in X. Then $Cl_X(J) = Cl_Y(J)$ where Cl_X is the closure in the Banach space X.

Now we consider case (iii) of (2). Let J be an invariant set in $L^P \times L^P$. Define $C(n) = e^{-Bn\omega}$ and $U(n) = U(t_0 + n\omega, t_0)$. Then $T^n = C(n) + U(n)$. Since T is a conditional α-contraction, J is precompact in $L^P \times L^P$. $\bigcup\limits_{n=0}^{\infty} U(n)J$ remains bounded in $\overset{*}{W}^{2,P} \times \overset{*}{W}^{2,P}$, so for some $p_1 > p$, $\bigcup\limits_{n=0}^{\infty} U(n)J$ is precompact in $L^r \times L^r$ for $p \le r \le p_1$. But $J = C(n)J + U(n)J$ and $C(n)J \to 0$ as $n \to \infty$. Thus, $J \subset Cl \ X(\bigcup\limits_{n=0}^{\infty} U(n)J)$. Hence, $J \subset L^r \times L^r$ by Lemma 4. Proceeding by induction we get J in a precompact invariant set in $L^q \times L^q$ for all $1 \le q \le \infty$. Since $\bigcup\limits_{n=0}^{\infty} U(n)J$ is precompact in $\overset{*}{W}^{1,P} \times \overset{*}{W}^{1,P}$ also, we can get the same result for $\overset{*}{W}^{1,q} \times \overset{*}{W}^{1,q}$ for all $1 \le q \le \infty$. Using this we may extend to $\overset{*}{W}^{2,q} \times \overset{*}{W}^{2,q}$ for all $1 \le q \le \infty$, and even to $C^2 \times C^2$.

4. <u>Limiting Behavior</u>. We will once again follow the format of previous sections. We will state a general theorem, indicate the methods used to prove the result, state several corollaries, and then apply the result to an example.

<u>Theorem 5</u>: <u>Let</u> X <u>and</u> Y <u>be admissible spaces</u> $(X, Y \in \mathscr{A})$ <u>and</u> $Y \hookrightarrow X$. <u>Suppose a bounded set in</u> X <u>dissipates points in</u> Y <u>under the period map</u> T <u>of</u> (2). <u>Then</u> T <u>is compact dissipative in</u> Y. <u>Furthermore, if</u> $Y \hookrightarrow\hookrightarrow X$ (<u>compactly imbedded</u>), <u>then</u> T <u>is bounded dissipative in</u> Y.

The proof uses the Sobolev imbedding theorem and the following results from [11], [13].

<u>Theorem 6</u>: <u>Let</u> $i: X_1 \hookrightarrow X_2$ <u>be a compact imbedding where</u> X_i <u>are Banach spaces topologies denoted</u> $\mathscr{F}_i$. <u>Let</u> $T, C,$ <u>and</u> U <u>be continuous operators mapping</u> X_i <u>into itself</u>, $i = 1, 2$. <u>Let</u> $T = C + U$ <u>with</u> C <u>a contraction in both spaces and</u> U <u>satisfying the following property: if</u> B <u>and</u> $U(B)$ <u>are bounded subsets</u>

of $(X_1, \mathscr{F}_2)$ then $U(B)$ is bounded in $(X_1, \mathscr{F}_1)$. Let $C(0) = 0$. Then point dissipative and compact dissipative in X_2 are equivalent. Also, the following three statements are equivalent: (i) There is a bounded set in X_2 which dissipates points in X_1, (ii) T is point dissipative in X_1, and (iii) T is bounded dissipative in X_1.

The following well-known theorem indicates the significance of these results.

Theorem 7: If T is a conditional α-contraction and compact dissipative in X then there is a maximal compact invariant set in X which is uniformly asymptotically stable, attracts neighborhoods of compact sets, and has a fixed point (corresponding to a periodic orbit of the equation). If, in addition, T is bounded dissipative, the maximal compact invariant set attracts bounded sets.

We now apply all these results to case (iii) of (2). Let T be point dissipative in $L^p \times L^p$ (or a weaker condition is to let a bounded set in $L^r \times L^r$ dissipate points in $L^p \times L^p$ for $1 \leq \gamma \leq p$) with $p \leq 1$. Then T is compact dissipative in the following class of spaces, which we will call $\mathscr{A}I$: $L^q \times L^q$, $\overset{*}{W}{}^{1,p} \times L^q$, $\overset{*}{W}{}^{2,q} \times L^q$, $\overset{*}{C} \times \overset{*}{C}$, $\overset{*}{C}{}^{\mu} \times \overset{*}{C}$, $\overset{*}{C}{}^{1} \times \overset{*}{C}$, $\overset{*}{C}{}^{1,\mu} \times \overset{*}{C}$, and $\overset{*}{C}{}^{2} \times \overset{*}{C}$, for $p \leq q \leq \infty$ and $0 < \mu < 1$. Also, T is bounded dissipative in class $\mathscr{A}II$ which consists of the following spaces: $\overset{*}{W}{}^{1,q} \times W^{1,q}$, $W^{2,q} \times W^{1,q}$, $W^{2,q} \times W^{2,q}$, $C^{\mu} \times C^{\nu}$, $C^1 \times C^{\beta}$, $C^{1,\alpha} \times C^{\beta}$, $C^2 \times C^{\beta}$, $C^1 \times C^1$, $C^{1,\alpha} \times C^1$, $C^2 \times C^1$, $C^{1,\mu} \times C^{1,\nu}$, $C^2 \times C^{1,\beta}$, and $\overset{*}{C}{}^{2} \times \overset{*}{C}{}^{2}$, where $q \geq p$, $0 < \beta, \alpha < 1$, and $0 < \nu \leq \mu < 1$. In addition, there is a set K which is the maximal compact invariant set for each admissible space, and in classes $\mathscr{A}I$ and $\mathscr{A}II$ it is uniformly asymptotically stable and attracts neighborhoods of compact sets in class $\mathscr{A}I$ and bounded sets in $\mathscr{A}II$. Furthermore, T has a fixed point (corresponding to a periodic orbit).

The proofs follow from the fact that an invariant set in one of the spaces must

be a precompact invariant set in all the spaces. Hence a maximal compact invariant set in one space becomes a maximal compact invariant space in all of the spaces. The attractivity properties follow from Theorem 7, and the original dissipative results follow from Theorem 6, the same inductive process, and repeated application of the Sobolev Imbedding Theorem.

In applications, it is often possible to prove boundedness or dissipative properties in some special space; for example, Lyapunov functions are often employed. But the same techniques cannot be applied to obtain boundedness or dissipative in any other spaces. The above results allow one to extend the boundedness and dissipative properties to many other spaces, along with the existence, very strong stability and attractivity properties of the maximal compact invariant set.

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